Published by Institute of Physics Publishing for SISSA/ISAS
Received: *September 17, 2002*
Accepted: *October 24, 2002*

The cosmological constant and oscillating metrics

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Abstract: The presence of a cosmological constant, Λ, in an action with higher powers of the curvature can produce rapidly oscillating metrics. We develop a perturbative approach for generating periodic solutions to the non-linear field equations for such actions based on a small amplitude expansion. We find that these oscillations have an amplitude proportional to $\sqrt{\Lambda}$ and a frequency of order the Planck mass. In a $4+1$-dimensional scenario, a family of metrics exists that are periodic in the extra dimension and are parameterized by an effective four-dimensional cosmological constant which drives a rapid oscillation.

Keywords: Field Theories in Higher Dimensions, Classical Theories of Gravity, Physics of the Early Universe.

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1. Introduction

Understanding the unnaturally small size of the cosmological constant poses one of severest challenges for a theory of gravity [1]. At late times and for large distances, the apparent size of the cosmological constant is constrained to be extremely small in terms of the natural scale for gravity, the Planck mass. In contrast, no observations bound the value of the cosmological constant during the earliest stages of the universe, when corrections to the Einstein-Hilbert action were non-negligible, and its presence can lead to a richer family of metrics. Among the solutions for a more general gravitational action, the presence of a positive cosmological constant does not inevitably lead to a de Sitter expansion. Such solutions must still yield or evolve into a low energy theory in which the *effective* cosmological constant is small to be phenomenologically acceptable. If the characteristic scales on which these metrics vary are of extremely high energy or short distance, then it may be possible to integrate out such features to arrive at a slowly varying effective theory.

To determine whether an action for gravity, generalized beyond an Einstein-Hilbert term, admits these features — natural coefficients for the terms in the action and a rapid variation — we must first solve the highly non-linear field equations. This task is difficult even when only the next curvature corrections are added. In $3+1$ dimensions, Horowitz and Wald [2] and later Starobinsky [3] discovered oscillating solutions for actions that included quadratic curvature terms but no cosmological constant. Numerical solutions were found in $4+1$ dimensions [4] in the presence of a cosmological constant and a scalar field, along with the quadratic curvature terms. In this latter scenario, metrics exist that depend periodically on the extra spatial coordinate so that choosing the size of the extra dimension to be the period produces a compact extra dimension without any fine-tunings or singularities. The parameters in the action fix the size of the extra dimension uniquely. However, without an analytical approach it becomes difficult to generalize these solutions to include an evolution in time. Without this freedom, it is difficult to understand how a universe starting from a more general state can find itself in one of these configurations.

In this article, we introduce a perturbative method for finding metrics with a periodic dependence on one or more coordinates. The perturbative parameter in this approach corresponds to the amplitude of the oscillation, which is generically of order $\sqrt{\Lambda}/M$, where Λ is the cosmological constant and M is the Planck mass. While this perturbative approach complements the numerical analysis of the static metrics in [4], it also allows the time-dependent extension of those metrics to be explored, which is too complicated for a numerical analysis. Moreover, this analytic, small amplitude expansion allows us to extract the dependence of the properties of the solution on the fundamental parameters in the action. In particular, we find the leading behavior for the amplitude and frequency of the oscillations in terms of the cosmological constant and coefficients of the four derivative curvature invariants.

These oscillating solutions are useful in $4+1$ dimensions since they provide an alternative to a purely de Sitter expansion when the cosmological constant in the full action is not exponentially small. In this scenario, the cosmological constant plays a dual role — both driving a rapid, Planck-frequency oscillation as well as supporting a compact extra dimension. As noted in [2] such oscillations would need to be damped out by some early stage in the universe — only an extremely small amplitude is allowed today [5]. The important property of these solutions is that the relative role of the cosmological constant in supporting the oscillations and the compactness is not fixed by the action. Unlike a de Sitter universe where a natural value for the cosmological constant is directly related to an extremely rapid expansion, in the oscillating scenario, this relative freedom does not preclude a large initial oscillation from subsequently relaxing, either by particle production or by interactions with charged particles.

In section 2 we introduce the small amplitude expansion in $3+1$ and $4+1$ dimensions for metrics that only depend on time. This setting avoids the complications of two variable solutions, yet illustrates some of the general properties of the oscillating solutions. We also briefly review the solution with a time independent metric in $4+1$ dimensions that depends periodically on the fifth dimension. Section 3 then generalizes this compact $4+1$-dimensional universe to include a rapidly oscillating time dependence. In section 4, we briefly discuss the compatibility of these solutions with an ordinary cosmology in the large dimensions.

2. Periodic metrics in the small amplitude limit

2.1 Oscillating metrics in $3+1$ dimensions

Periodic metrics exist in any number of dimensions when we generalize the standard Einstein-Hilbert action to include not only a cosmological constant but higher derivative corrections as well. For simplicity, we first show the existence of an oscillating metric

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in $3+1$ dimensions. We begin with a gravitational action with arbitrary powers of the curvature,[1]

$$S = M^2 \int d^4x \sqrt{-g} \left(-2\Lambda + R + aR^2 + bR_{\mu\nu}R^{\mu\nu} + cR_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} + \cdots \right) + \tag{2.1}$$
$$+ M^2 \int d^4x \sqrt{-g} \mathcal{L} \,,$$

and consider solutions of the form,

$$ds^2 = -dt^2 + e^{B(t)} g_{ij} dx^i dx^j \,. \tag{2.2}$$

We assume that $\mathcal{L}$ contains a free massless scalar field, $\phi(t)$, and in particular contributes to the energy-momentum tensor as a perfect fluid with density, ρ, and pressure, p, of $\rho = p = \frac{1}{4}\dot{\phi}^2 + C$. C is some additional constant contribution, perhaps arising as a self-consistent Casimir effect.

We shall show the existence of solutions where the function $B(t)$ is a smooth, periodic function of time. When the amplitude, ϵ, of these oscillations is small, this dimensionless parameter allows us to solve the non-linear field equations for $B(t)$ pertubatively. For a consistent expansion, we must also specify how the sources of gravity — the cosmological constant, scalar field and Casimir contribution — scale with ϵ. In the solutions we have found, ΛM^{-2}, $\frac{1}{4}\dot{\phi}^2 M^{-4}$, and CM^{-4}, are all of order ϵ^2. In the following, we shall not write factors of the Planck mass explicitly, absorbing M^{-2} factors into $\dot{\phi}^2$, C, ρ, p, etc.

At linear order in ϵ the field equations reduce to one equation,

$$\frac{d^2B}{dt^2} + 2\nu \frac{d^4B}{dt^4} + \sum_{k=3}^{\infty} \sum_i \nu_{k,i} \frac{d^{2k}B}{dt^{2k}} = 0 \,, \tag{2.3}$$

where $\nu = 3a+b+c$ and the $\nu_{k,i}$ are various combinations of coefficients of terms higher order in R. This equation has sinusoidal solutions $B(t) = \epsilon \cos(\omega t)$ as long as the polynomial

$$-\omega^2 + 2\nu\,\omega^4 + \sum_{k=3}^{\infty} \sum_i (-1)^k \nu_{k,i} \omega^{2k} = 0 \tag{2.4}$$

has at least one real root. Thus for a range of parameters in the original action — so that no fine tuning is needed — such solutions exist. The non-linearity of the field equations appears in the next order terms in the expansion where the amplitude ϵ is determined in terms of the cosmological constant and other parameters in the action.

This behavior indicates that, at least in the small amplitude limit, periodic metrics of the form (2.2) exist to an arbitrary order in the derivative expansion. This derivative expansion and its associated solutions should arise from a general theory as we near the Planck scale.

[1] Our convention for the signature of the metric is $(-,+,\ldots,+)$ while the Riemann curvature tensor is defined by $-R^{\kappa}{}_{\lambda\mu\nu} \equiv \partial_\nu \Gamma^{\kappa}_{\lambda\mu} - \partial_\mu \Gamma^{\kappa}_{\lambda\nu} + \Gamma^{\kappa}_{\rho\nu}\Gamma^{\rho}_{\lambda\mu} - \Gamma^{\kappa}_{\rho\mu}\Gamma^{\rho}_{\lambda\nu}$. Our sign convention for Λ has changed from [4].

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To obtain more explicit results we shall hereafter drop the terms in the action of order R^3 and higher. It should be clear that the existence of the solutions does not depend on this truncation. While the quantitative results certainly are sensitive to the truncation, the qualitative picture should not be altered. Since we are not expanding in powers of small derivatives, the higher derivative terms in the equations are just as important as lower derivative terms so that the usual procedure of using equations of motion at low order in a derivative expansion to simplify the analysis at higher order does not apply here.

After the truncation we find

$$\omega = \frac{1}{\sqrt{2\nu}} \tag{2.5}$$

so that ν should satisfy the constraint $\nu > 0$. When we extend to order ϵ^2, the analogue of (2.3) is a non-linear differential equation whose solution is

$$B(t) = \epsilon \cos(\omega t) - \frac{3}{16}\epsilon^2 \cos(2\omega t)\,, \tag{2.6}$$

with

$$\epsilon^2 = -\frac{16}{3}\Lambda\nu\,, \qquad \text{and} \qquad \dot{\phi}^2 + 4C = 4\Lambda\,. \tag{2.7}$$

This result shows the importance of the cosmological constant for the existence of an oscillating solution since the amplitude is proportional to $\sqrt{-\Lambda}$. Since $\nu > 0$ we also see that $\Lambda < 0$. Expanding to order ϵ^3 we have

$$\begin{aligned}
B(t) &= \epsilon \cos(\omega t) - \frac{3}{16}\epsilon^2 \cos(2\omega t) + \frac{13}{256}\epsilon^3 \cos(3\omega t)\,,\\
\omega &= (2\nu)^{-1/2}\left(1 - \frac{45}{64}\epsilon^2\right),\\
\dot{\phi}^2 + 4C &= 4\Lambda\left(1 - 3\epsilon\cos(\omega t)\right).
\end{aligned} \tag{2.8}$$

Only at this order does a time dependence of $\dot{\phi}$ appear, showing the necessity of a dynamical field in addition to gravity. Note that the correction to the frequency is of order ϵ^2 rather than ϵ. A numerical analysis [4] of the full equations shows that there really is an exact solution that is being approximated here and such a solution exists as long as $0 < -\nu\Lambda < 1/24$.

We see the first instance of a generic feature of oscillating metric solutions, that the square of the oscillation amplitude ϵ is proportional to the cosmological constant. The C parameter introduces a puzzle, as can be seen by considering the leading order ϵ^2 contributions to the energy-momentum tensor,

$$\begin{aligned}
\rho &= \Lambda + \frac{1}{4}\dot{\phi}^2 + C = 2\Lambda < 0\,,\\
p &= -\Lambda + \frac{1}{4}\dot{\phi}^2 + C = 0\,.
\end{aligned} \tag{2.9}$$

Such a density and pressure violates various positive energy conditions [6] so that it is difficult to see how the required negative C can arise. While this apparent difficulty might result from our truncation of the action beyond the R^2 order, we do not pursue this possibility here.

2.2 Oscillating metrics in $4+1$ dimensions

By extending to $4+1$ dimensions, we now have the possibility that the behavior of the oscillations in the extra dimension differs from that in the large dimensions,

$$ds^2 = -dt^2 + e^{B(t)}\delta_{ij}dx^i dx^j + e^{C(t)}dy^2 \,. \tag{2.10}$$

At first order in ϵ_t, using[2]

$$B(t) = \epsilon_t \cos(\omega_t t)\,, \qquad C(t) = \epsilon_t \eta \cos(\omega_t t) \tag{2.11}$$

two solutions to the field equations exist,

$$\eta = 1\,, \qquad \omega_t = \sqrt{\frac{3}{\mu}}\,, \qquad \text{and} \qquad \eta = -3\,, \qquad \omega_t = \frac{1}{\sqrt{3\mu - 16\nu}}\,. \tag{2.12}$$

We have chosen new linear combinations of the coefficients of the R^2 terms in the five-dimensional action,

$$\begin{aligned} \mu &= 16a + 5b + 4c\,, \\ \nu &= 3a + b + c\,, \\ \lambda &= 5a + b + \frac{1}{2}c\,. \end{aligned} \tag{2.13}$$

ν is the coefficient of the squared Weyl term. The coefficient of the Gauss-Bonnet term, λ, does not appear until order ϵ_t^3.

The first solution is completely analogous to the previous case and it leads to the same problem. Therefore let us consider the second solution and continue the expansion to order ϵ_t^3,

$$\begin{aligned} B(t) &= \epsilon_t \cos(\omega_t t) + b_2\epsilon_t^2 \cos(2\omega_t t) + b_3\epsilon_t^3 \cos(3\omega_t t)\,, \\ C(t) &= \epsilon_t\eta \cos(\omega_t t) + c_2\epsilon_t^2 \cos(2\omega_t t) + c_3\epsilon_t^3 \cos(3\omega_t t)\,, \\ \frac{1}{4}\dot{\phi}^2 &= \Lambda\,, \end{aligned} \tag{2.14}$$

where the frequency and and amplitudes are

$$\begin{aligned} \omega_t &= \frac{1}{\sqrt{3\mu - 16\nu}}\left(1 + \omega_{t,02}\epsilon_t^2\right), \\ \eta &= -3\left(1 + \eta_3\epsilon_t^2\right), \\ \epsilon_t^2 &= \frac{4}{3}\Lambda(3\mu - 16\nu)\,, \end{aligned} \tag{2.15}$$

and where

$$b_2 = c_2 = -\frac{3}{4}\frac{8\nu - \mu}{48\nu - 5\mu} \tag{2.16}$$

with b_3, c_3, $\omega_{t,02}$, and η_3 as given in the appendix. Here Λ denotes the five-dimensional cosmological constant. We see that $\dot{\phi}^2$ remains constant at this order, unlike the previous

[2] We introduce subscripts to distinguish ϵ_t from ϵ_y used in the next subsection.

case, although this behavior does not continue to hold at higher orders. More importantly, from the results for ω_t and ϵ_t^2 we see that $3\mu - 16\nu$ and Λ must both be positive, and thus we get a consistent result for $\dot{\phi}^2$ without any additional Casimir contribution. The pressure and density which support this geometry are $\rho = 2\Lambda > 0$ and $p = p_y = 0$. The change of sign for the energy density has occurred through the introduction of an extra dimension which oscillates with a different amplitude than the other spatial dimensions.

We might hope that since the cosmological constant is related to the amplitude of an oscillating field, it could relax to zero with the decay of this field. But in this scenario the amplitude is directly related to Λ, a fundamental constant in the action, and cannot relax. This observation suggests that a better approach would be to find a family of metrics in which some *effective* four-dimensional cosmological constant, still related to an oscillating field, exists. In such a scenario, this effective cosmological constant is not forbidden from relaxing while the fundamental Λ does not change.

2.3 Metrics periodic in an extra dimension

Before examining metrics with these properties, we describe the case of a static 4 + 1-dimensional universe where a periodic metric produces a naturally compact extra dimension. Consider a metric of the form

$$ds^2 = e^{A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \,. \tag{2.17}$$

where $A(y)$ is periodic and y coordinates separated by one period are identified. In [4], numerical solutions of this form were studied in detail. In this scenario the size of the compact space, the period, is dynamically determined, and there is no radion stabilization problem. Rescaling $dy^2 \to \kappa^2 dy^2$ has no physical significance since it would result in the period of $A(y)$ being scaled by κ^{-1}. The physical size of the compact space, the product of scale factor and range of y, remains the same.

With a scalar field $\phi(y)$ we consider a contribution to the energy-momentum tensor of the form $\rho = -p = p_y = \frac{1}{4}\phi'^2 + C$. We again introduce a constant C as a possible Casimir effect, and shall return to its role later. The scalar field $\phi(y)$ is also compact, with the solution determining the range over which it varies. Thus we have a dynamical generation of a compact internal space, in parallel with the dynamical generation of a compact fifth dimension. No fine-tuning of parameters in the action is required for either [4].

The solution to order ϵ_y^3 for the metric and the scalar field is

$$\begin{aligned} A(y) &= \epsilon_y \cos(\omega_y y) - \frac{1}{4}\epsilon_y^2 \cos(2\omega_y y) + \frac{1}{144}\frac{13\mu - \lambda}{\mu}\epsilon_y^3 \cos(3\omega_y y)\,, \\ \phi'^2 + 4C &= -4\Lambda\left(1 - 4\epsilon_y \cos(\omega_y y)\right), \end{aligned} \tag{2.18}$$

where the frequency and the amplitude are given by

$$\omega_y = \sqrt{\frac{-3}{\mu}}\left(1 + \frac{1}{4}\frac{\lambda - 5\mu}{\mu}\epsilon_y^2\right), \qquad \epsilon_y^2 = -\frac{4}{9}\Lambda\mu\,. \tag{2.19}$$

The expressions for ω_y and ϵ_y^2 indicate that μ must be negative and Λ must be positive.

The leading contributions to the energy-momentum tensor yield $\rho = p = 0$, $p_y = -2\Lambda$. Thus although the action has a positive five dimensional Λ, the *effective* four-dimensional cosmological constant vanishes.

Not surprisingly, it is also true that there are other solutions in which this effective cosmological constant is nonvanishing [8]. The apparent cosmological constant seen by a low-energy observer is adjustable, depending upon the warping of the fifth dimension. The problem here is that we don't know why or how the system would relax to the flat low-energy solution.

It is unclear whether a negative pressure in a fifth dimension should be considered unphysical, but it is less of a concern for the following reason. Even though a small five-dimensional Λ is convenient for our expansion, there is no reason that it should be small in Planck units. When Λ is not small and a higher derivative scalar term is included there are periodic solutions that do not require the Casimir contribution, C [4]. The required energy-momentum tensor that supports this geometry, which is in that case strongly y dependent, arises explicitly from the scalar field configuration.

3. Oscillating metrics with a warped extra dimension

A universe with only a rapid time oscillation encounters the same difficulty that occurs for a de Sitter solution — the amplitude of the former and the expansion rate of the latter are both proportional to $\sqrt{\Lambda}$ so that no direct method for the relaxation of either feature is available and we must simply assume from the beginning an exponentially small size for Λ in its natural units. In the static, warped scenarios described in the previous subsection we do not need to make any such fine-tuning of Λ to obtain a universe that appears approximately flat at low energies. However, this solution apparently involves tuning the initial conditions so that no effective, low-energy cosmological constant is present. In this section we shall study metrics that depend periodically on both time and the extra dimension. The cosmological constant then serves partially to drive the oscillations and partially to support the compact dimension. The oscillations are thereby no longer forbidden from damping away since their amplitude is given by a free parameter of the solution and not $\sqrt{\Lambda}$.

Generalizing the metrics (2.10) and (2.17) to first order in the amplitudes in the two oscillations, ϵ_t and ϵ_y, we consider a metric of the form

$$ds^2 = -e^{A(y)}dt^2 + e^{A(y)+B(t)}\delta_{ij}dx^i dx^j + e^{C(t)}dy^2 \,, \tag{3.1}$$

with

$$A(y) = \epsilon_y \cos(\omega_y y)\,, \qquad B(t) = \epsilon_t \cos(\omega_t t)\,, \qquad C(t) = -3\epsilon_t \cos(\omega_t t)\,. \tag{3.2}$$

We have already seen that

$$\omega_y = \sqrt{\frac{-3}{\mu}}\,, \qquad \text{and} \qquad \omega_t = \frac{1}{\sqrt{3\mu - 16\nu}}\,. \tag{3.3}$$

The range of y is determined by ω_y, but the size of the compact space oscillates in time with frequency ω_t.

At next order in ϵ_t and ϵ_y the nonlinear field equations induce a mixing between the t and y dependent terms, and thus the metric assumes a more complicated form,

$$ds^2 = -e^{A(y)+E(y,t)}dt^2 + e^{A(y)+B(t)+F(y,t)}\delta_{ij}dx^i dx^j + e^{C(t)+G(y,t)}dy^2 \,. \tag{3.4}$$

At second order we find

$$\begin{aligned}
A(y) &= \epsilon_y \cos(\omega_y y) - \frac{1}{4}\epsilon_y^2 \cos(2\omega_y y)\,, \\
B(t) &= \epsilon_t \cos(\omega_t t) + b_2 \epsilon_t^2 \cos(2\omega_t t)\,, \\
C(t) &= -3\epsilon_t \cos(\omega_t t) + c_2 \epsilon_t^2 \cos(2\omega_t t)\,, \\
E(y,t) &= e_{11}\epsilon_y \epsilon_t \cos(\omega_y y)\cos(\omega_t t)\,, \\
F(y,t) &= f_{11}\epsilon_y \epsilon_t \cos(\omega_y y)\cos(\omega_t t)\,, \\
G(y,t) &= g_{11}\epsilon_y \epsilon_t \cos(\omega_y y)\cos(\omega_t t)\,.
\end{aligned} \tag{3.5}$$

b_2, and c_2 are as before, f_{11} and g_{11} are given in the appendix, and e_{11} remains undetermined.[3] We find that

$$\begin{aligned}
\epsilon_y^2 &= \frac{4}{3}\frac{\Lambda}{\omega_y^2} - \frac{\omega_t^2}{\omega_y^2}\epsilon_t^2\,, \\
\phi'^2 + 4C &= -4\Lambda + 3\omega_t^2\epsilon_t^2\,, \\
\dot{\phi}^2 &= 3\omega_t^2\epsilon_t^2\,,
\end{aligned} \tag{3.6}$$

where ω_y and ω_t are the first order values in (3.3).

The significant feature of this solution is that the cosmological constant is related to *both* amplitudes — we can freely vary ϵ_t, for example, with a fixed value of Λ, and still find a solution since ϵ_y can compensate for the changes in the time oscillations. To emphasize the similarity of this component of the solution and the behavior we saw in the previous section, where a cosmological constant drove an oscillating metric in $3+1$ dimensions, we introduce a new positive constant, $\Lambda_{\rm osc}$,

$$\Lambda_{\rm osc} \equiv \Lambda + C + \frac{1}{4}\phi'^2 = \frac{3}{4}\omega_t^2\epsilon_t^2\,. \tag{3.7}$$

Note that $\Lambda_{\rm osc}$ only parameterizes the size of the time oscillations in the metric and is not fixed by any quantity in the action. In the limit $\Lambda_{\rm osc} \to 0$, we recover the static metric of eq. (2.17).

In terms of this new parameter, the density and pressure which support this geometry have the form

$$\begin{aligned}
\rho &= \Lambda_{\rm osc} + \frac{1}{4}\dot{\phi}^2 = 2\Lambda_{\rm osc} > 0\,, \\
p &= -\Lambda_{\rm osc} + \frac{1}{4}\dot{\phi}^2 = 0
\end{aligned} \tag{3.8}$$

[3] The value of e_{11} does not affect five-dimensional curvature invariants at this order, and its value, given in the appendix, is determined at third order in the expansion.

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in the large directions. $\Lambda_{\rm osc}$ appears here rather than the five-dimensional Λ. The latter drives the warping of the fifth dimension and appears in the pressure in the extra dimension, $p_y = -2\Lambda + 2\Lambda_{\rm osc}$. Note that since the action contains terms quadratic in the curvature tensors, the energy-momentum tensor is not conventionally related to the Einstein tensor, $G_{\mu\nu}$. For example, when averaging over the fifth dimension we find

$$G_{tt} = -2\Lambda_{\rm osc}\sin^2(\omega_t t)\,. \tag{3.9}$$

This equation could perhaps be interpreted as an effective energy density obtained by combining the contributions from the R^2 terms together with the actual contributions to the energy-momentum tensor.

In principle, we can continue the small amplitude expansion to higher orders although the expressions for the new coefficients quickly become quite lengthy. At third order, the expressions for the frequencies receive small corrections,

$$\begin{aligned} \omega_t &= \frac{1}{\sqrt{3\mu - 16\nu}}\left(1 + \omega_{t,02}\epsilon_t^2 + \omega_{t,20}\epsilon_y^2\right), \\ \omega_y &= \sqrt{\frac{-3}{\mu}}\left(1 + \frac{1}{4}\frac{\lambda - 5\mu}{\mu}\epsilon_y^2 + \omega_{y,02}\,\epsilon_t^2\right). \end{aligned} \tag{3.10}$$

The third order corrections involving terms with purely t or y dependence are as in the previous sections. η is as in (2.15), the right side of the second equation in (3.6) is multiplied by the same $(1 - 4\epsilon_y\cos(\omega_y y))$ factor appearing in (2.18), and the third equation in (3.6) is not corrected.[4] The third order corrections involving a product of a t or a y dependence are contained in the following expansions:

$$\begin{aligned} F(y,t) &= f_{11}\epsilon_y\epsilon_t\cos(\omega_y y)\cos(\omega_t t) + f_{21}\epsilon_y^2\epsilon_t\cos(2\omega_y y)\cos(\omega_t t) + \\ &\quad + f_{12}\epsilon_y\epsilon_t^2\cos(\omega_y y)\cos(2\omega_t t)\,, \\ G(y,t) &= g_{11}\epsilon_y\epsilon_t\cos(\omega_y y)\cos(\omega_t t) + g_{21}\epsilon_y^2\epsilon_t\cos(2\omega_y y)\cos(\omega_t t) + \\ &\quad + g_{12}\epsilon_y\epsilon_t^2\cos(\omega_y y)\cos(2\omega_t t) + g'_{21}\epsilon_y^2\epsilon_t\cos(\omega_t t)\,. \end{aligned} \tag{3.11}$$

The expressions for the coefficients in (3.10) and (3.11) are given in the appendix.

4. Evolution and cosmology

The previous section introduced a class of metrics that are periodic in both time and an extra dimension in a universe with a non-finely tuned cosmological constant. The existence of a small, compact dimension will not be apparent to observers who can only probe length scales much larger than $\sqrt{-\mu}$. In contrast, the existence of a Planck-frequency oscillation would not be compatible with current observations [2] unless its amplitude is extremely small [5]. Since the time oscillations are driven by a positive contribution to the energy density (3.8) the oscillating metric in (3.4) would seem to be unstable against relaxing

[4] These results are consistent with $\phi(t,y) = \phi_1(t) + \phi_2(y)$, but alternatively there could be two independent scalar fields with dependence only on t and y, respectively.

toward a metric (2.17) which is flat in the large dimensions. To investigate the evolution of the oscillation amplitude and its dynamical coupling to the cosmological expansion would require the addition of some time dependence to the amplitudes, $\epsilon_t \to \epsilon_t(t)$ and $\epsilon_y \to \epsilon_y(t)$. However, analyzing this more general time dependence becomes intractable for the above analytic expansion. In this section we shall restrict therefore to showing how this picture differs from a Kaluza-Klein model with a flat extra dimension and to understanding how a rapid oscillation of *constant* amplitude is incompatible with a cosmological expansion.

When the large dimensions evolve with a Robertson-Walker expansion — assuming the oscillations have decayed — this scenario behaves very differently from a standard Kaluza-Klein cosmology in which two time dependent scale factors govern the evolution of the size of the large spatial dimensions and the compact space, respectively. In this latter case, one must ensure that the resulting time evolution of physical constants after integrating out the extra dimensions is sufficiently small [9]. In contrast, as described in section 3, the second scale factor in the warped scenario is not an independent dynamical quantity since the size of the compact space is already determined. This property makes a standard cosmological evolution more natural. For example a power law scale factor $a(t) = t^\alpha$ for the three large spatial dimensions is determined as usual by

$$\alpha = \frac{2}{3}\frac{1}{1+w}\,, \tag{4.1}$$

where $p = w\rho$. For a given w the time dependence of ρ and p is determined, as is the matter contribution to the five-dimensional pressure,

$$p_y|_{\text{matter}} = \frac{1}{2}(3w-1)\rho\,. \tag{4.2}$$

This result follows directly from the five-dimensional Einstein equation given that the compact dimension has fixed size. When the variation in the non-compact coordinates is small compared to the size of the compact dimension, the contribution of ordinary matter and radiation fields to the pressure in the extra dimension will also be small compared to the order ϵ_y contribution to p_y; the latter is -2Λ which drives the warped compactification in the first place.

If we attempt to introduce cosmological expansion with a scale factor $a(t)$ along with a rapid oscillation of constant amplitude,

$$ds^2 = e^{A(y)}\left(-dt^2 + e^{\epsilon_t\cos(\omega_t t)+\cdots}a^2(t)\delta_{ij}dx^i dx^j\right) + e^{-3\epsilon_t\cos(\omega_t t)+\cdots}dy^2\,, \tag{4.3}$$

we should apparently choose $B(t) = \epsilon_t\cos(\omega_t t) + f(t)$, where $f(t) \equiv 2\ln(a(t))$. However, inserting this metric into the leading small amplitude field equations produces crossterms of the form $\dot f(t)\sin(\omega_t t)$ which do not cancel. Moreover, they cannot be canceled by ordinary matter effects because we are assuming that the latter is a long wavelength effect. We can eliminate such terms and thereby find solutions for the *gravitational* field equations if we include the following dependence on $f(t)$ in $B(t)$ and $C(t)$:

$$\begin{aligned}
B(t) &= \epsilon_t\cos(\omega_t t) + f(t) + b_4\epsilon_t\dot f(t)\omega_t^{-1} f\sin(\omega_t t)\,,\\
C(t) &= -3\epsilon_t\cos(\omega_t t) + c_4\epsilon_t\dot f(t)\omega_t^{-1}\sin(\omega_t t)\,,\\
\dot\phi^2(t) &= 3\omega_t^2\epsilon_t^2 + 6\epsilon_t\dot f(t)\omega_t\sin(\omega_t t)
\end{aligned} \tag{4.4}$$

with

$$b_4 = \frac{2}{3}\frac{\omega_t^2}{\omega_y^2}\frac{1}{\epsilon_y^2} + \frac{3\omega_y^2 - \omega_t^2}{2\omega_y^2 + 2\omega_t^2}\,, \qquad c_4 = -2\frac{\omega_t^2}{\omega_y^2}\frac{1}{\epsilon_y^2}\,. \tag{4.5}$$

Defining $\dot{f}(t) \approx \epsilon_f M$, the size of the b_4 and c_4 terms are of order $\epsilon_t\epsilon_f\epsilon_y^{-2}$ and the expansion appears to make sense for $B(t)$ and $C(t)$ in the regime of interest, $\epsilon_t \approx \epsilon_f \ll \epsilon_y$.

The incompatibility of the cosmological expansion with the oscillations arises in the expression for $\dot{\phi}^2(t)$, where we see that the correction is of the same order as the first term. The presence of this term does not allow the *scalar* field equation for $\phi(t)$ to be satisfied. This difficulty is presumably related to the fact that the small amplitude expansion does not account for a varying amplitude for the oscillations. This investigation is continued numerically in $3+1$ dimensions in [7].

5. Conclusions

In the very early universe, when higher order curvature invariants become important in the action, the presence of a cosmological constant can produce a rapidly oscillating metric rather than a de Sitter expansion. In a $3+1$-dimensional universe or a universe with a flat extra dimension, since the cosmological constant is directly related to the amplitude of these oscillations, no method for damping these oscillations is available. In a universe with a warped extra dimension, the cosmological constant can play a dual role both supporting a compact extra dimension and, as before, driving oscillations. Since the relative amount which the cosmological constant supports these two periodic behaviors is a free parameter, the oscillations can decay through their coupling to other fields present. In the resulting approximately static universe, the cosmological constant solely supports the compact dimension and so does not need to be small to produce an effectively flat $3+1$-dimensional universe in the low energy limit.

We have used this scenario as a illustration of how extra dimensions and a more general response to including a Λ in the action can evade the usual theorems forbidding the relaxation of a cosmological constant — since in fact Λ does not change and instead only an effective parameter damps away. Nevertheless, significantly more work is needed to apply this approach to understanding the cosmological constant problem.

The detailed results of this paper have relied on a truncation of the higher derivative terms to the R^2 order. Consideration of higher order terms may open up even more possibilities, but they should not qualitatively change the possibilities we have discussed. These possibilities could also exist for the true underlying theory for gravity.

The time-dependent oscillations have appeared as solutions disjoint from the usual exponentially expanding de Sitter solutions, since both types of time dependence cannot coexist in our version of the small amplitude expansion. It may be that a more general type of solution exists for which pure oscillations and a pure cosmological expansion exist as limiting cases. This possibility is pursued in $3+1$ dimensions in [7]. A nontrivial coupling between the amplitude of oscillations and the large scale expansion then leaves open the possibility that oscillations could exist even at late times.

JHEP10(2002)052

Acknowledgments

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada.

A. Higher order coefficients

In this appendix, we list some of the more cumbersome coefficients of the higher order terms in the various small amplitude expansions. In the solution to the five-dimensional metric with only time dependence (2.14) we have

$$b_3 = \frac{-15\lambda\mu^2 + 239\mu\lambda\nu + 30\mu^3 - 1018\nu\mu^2 + 9780\mu\nu^2 - 912\lambda\nu^2 - 27648\nu^3}{432\nu(224\nu\mu - 15\mu^2 - 768\nu^2)}, \tag{A.1}$$

$$c_3 = \frac{30\mu^3 - 15\lambda\mu^2 + 722\nu\mu^2 + 179\mu\lambda\nu - 15132\mu\nu^2 - 336\lambda\nu^2 + 55296\nu^3}{432\nu(224\nu\mu - 15\mu^2 - 768\nu^2)}, \tag{A.2}$$

$$\omega_{t,02} = \frac{97\mu^2 - 5\lambda\mu - 1452\nu\mu + 4992\nu^2 + 48\lambda\nu}{4(224\nu\mu - 15\mu^2 - 768\nu^2)}, \tag{A.3}$$

$$\eta_3 = \frac{-\lambda + 2\mu - 9\nu}{6(\mu - 6\nu)}. \tag{A.4}$$

In the metrics with both t and y dependence, at second order, the coefficients of the mixing terms (3.5) are

$$f_{11} = \frac{-13824\nu^2 + 5124\mu\nu - 32\lambda\mu + 192\lambda\nu - 477\mu^2}{6\mu(3\mu - 16\nu)}, \tag{A.5}$$

$$\begin{aligned} g_{11} &= -3e_2\frac{(3\mu - 16\nu)}{\mu} + \frac{221184\nu^3 + 24648\mu^2\nu - 128064\mu\nu^2}{2\mu^2(3\mu - 16\nu)} - \\ &\quad - \frac{3072\lambda\nu^2 + 112\lambda\mu^2 + 1575\mu^3 - 1152\nu\lambda\mu}{2\mu^2(3\mu - 16\nu)}. \end{aligned} \tag{A.6}$$

The third order corrections (3.10) and (3.11) have the following coefficients.

$$\begin{aligned} \omega_{y,02} &= \big(-1536\nu\lambda^2 - 80448\nu\lambda\mu - 783558\mu^2\nu + 256\mu\lambda^2 + 7344\lambda\mu^2 + 221184\lambda\nu^2 + \\ &\quad + 4326912\mu\nu^2 - 7962624\nu^3 + 47277\mu^3\big) \Big/ \big(288\mu(3\mu - 16\nu)^2\big), \end{aligned} \tag{A.7}$$

$$\begin{aligned} \omega_{t,20} &= -\big(50085\mu^3 - 814086\mu^2\nu + 7344\lambda\mu^2 - 80448\nu\lambda\mu + 256\mu\lambda^2 + 4409856\mu\nu^2 - \\ &\quad - 1536\nu\lambda^2 + 221184\lambda\nu^2 - 7962624\nu^3\big) \Big/ \big(96\mu^2(3\mu - 16\nu)\big), \end{aligned} \tag{A.8}$$

$$\begin{aligned} e_{11} &= -\big(15925248\nu^4 - 10215936\mu\nu^3 + 207360\mu\lambda\nu^2 + 6165\mu^4 + 4864\mu^3\lambda + \\ &\quad + 256\mu^2\lambda^2 - 218382\mu^3\nu - 1536\mu\nu\lambda^2 - 221184\lambda\nu^3 + 2344032\mu^2\nu^2 - \\ &\quad - 56640\mu^2\lambda\nu\big) \Big/ \big(72\mu(\mu - 6\nu)(3\mu - 16\nu)^2\big), \end{aligned} \tag{A.9}$$

$$\begin{aligned} f_{21} &= -\big(285273\mu^5 - 859963392\nu^5 + 135168\nu\mu^2\lambda^2 - 741888\mu\nu^2\lambda^2 + \\ &\quad + 65992320\mu\lambda\nu^3 - 7035288\mu^4\nu + 2479416\mu^3\lambda\nu + 1354752\nu^3\lambda^2 - \end{aligned} \tag{A.10}$$

$$
\begin{aligned}
&- 8192\mu^3\lambda^2 - 120780\mu^4\lambda + 69600906\mu^3\nu^2 - 19152288\mu^2\nu^2\lambda - 85598208\lambda\nu^4 + \\
&+ 860087808\mu\nu^4 - 345400308\mu^2\nu^3\Big) \Big/ \Big(144\mu(\mu-6\nu)(3\mu-16\nu)^2(7\mu-36\nu)\Big)\,,
\end{aligned}
$$

$$
\begin{aligned}
f_{12} = \Big(&-36720677376\mu^2\nu^5 + 199247877\mu^5\nu^2 - 11515515\mu^6\nu + 640\mu^5\lambda^2 + \\
&+ 29098\mu^6\lambda + 282591\mu^7 + 10818446400\mu^3\nu^4 - 55037657088\nu^7 + \\
&+ 1528823808\nu^6\lambda - 10616832\nu^5\lambda^2 - 25344\mu^4\nu\lambda^2 - 1901363418\mu^4\nu^3 - \\
&- 1132452\mu^5\lambda\nu - 146614464\mu^3\lambda\nu^3 + 17867184\mu^4\lambda\nu^2 + 68860772352\mu\nu^6 - \\
&- 1569964032\mu\nu^5\lambda - 2566656\mu^2\nu^3\lambda^2 + 662611968\mu^2\nu^4\lambda + 372480\mu^3\nu^2\lambda^2 + \\
&+ 8404992\nu^4\mu\lambda^2\Big) \Big/ \Big(72\mu^2(\mu-6\nu)(3\mu-16\nu)^2(\mu-4\nu)(5\mu-48\nu)\Big)\,, \qquad \text{(A.11)}
\end{aligned}
$$

$$
\begin{aligned}
g_{21} = -\Big(&7735005\mu^6 + 110075314176\nu^6 + 131254272\nu^3\lambda^2\mu + 4668928\mu^3\nu\lambda^2 + \\
&+ 4215635712\lambda\nu^3\mu^2 - 18316369656\mu^3\nu^3 - 10733617152\nu^4\lambda\mu + 81818128\mu^4\lambda\nu - \\
&- 37174272\nu^2\lambda^2\mu^2 + 2793157068\mu^4\nu^2 - 3233928\mu^5\lambda - 227538114\mu^5\nu + \qquad \text{(A.12)} \\
&+ 67677175296\nu^4\mu^2 + 10956570624\nu^5\lambda - 133596905472\nu^5\mu - 219392\mu^4\lambda^2 - \\
&- 173408256\nu^4\lambda^2 - 829677504\mu^3\lambda\nu^2\Big) \Big/ \Big(96\mu^2(\mu-6\nu)(3\mu-16\nu)^2(7\mu-36\nu)\Big)\,,
\end{aligned}
$$

$$
\begin{aligned}
g_{12} = \Big(&2332845\mu^8 - 5361075486720\mu\nu^7 - 97844723712\lambda\nu^7 + 70400\mu^6\lambda^2 + \\
&+ 3522410053632\nu^8 + 277363541376\mu^4\nu^4 - 37624542552\mu^5\nu^3 - \\
&- 132976434\mu^7\nu - 1262573733888\mu^3\nu^5 + 1351640\mu^7\lambda + \\
&+ 3486633984000\mu^2\nu^6 + 3049013340\mu^6\nu^2 - 45487152\mu^6\lambda\nu + \\
&+ 20788224\mu^4\nu^2\lambda^2 - 721944576\mu\nu^5\lambda^2 + 679477248\nu^6\lambda^2 + \\
&+ 646135872\mu^5\lambda\nu^2 - 1927680\mu^5\nu\lambda^2 - 5107325184\mu^4\lambda\nu^3 + \\
&+ 126977310720\mu\nu^6\lambda + 376012800\nu^4\mu^2\lambda^2 + 24644653056\mu^3\nu^4\lambda - \\
&- 116299776\mu^3\nu^3\lambda^2 - 73583493120\mu^2\nu^5\lambda\Big) \Big/ \\
&\Big/ \Big(384\mu^3(\mu-4\nu)(5\mu-48\nu)(\mu-6\nu)(3\mu-16\nu)^2\Big)\,, \qquad \text{(A.13)}
\end{aligned}
$$

$$
\begin{aligned}
g'_{21} = \Big(&9243\mu^4 + 5308416\nu^4 - 3072\nu^2\lambda^2 + 1335540\mu^2\nu^2 - 88704\mu\lambda\nu^2 + \\
&+ 1280\mu\nu\lambda^2 + 18032\mu^2\lambda\nu - 4359168\mu\nu^3 - 1244\mu^3\lambda - 128\mu^2\lambda^2 - \\
&- 181428\mu^3\nu + 147456\lambda\nu^3\Big) \Big/ \Big(4\mu^2(3\mu-16\nu)(\mu-6\nu)\Big)\,. \qquad \text{(A.14)}
\end{aligned}
$$

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